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UNITARY MEASURES ON LCA GROUPS

BY

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ABSTRACT. A unitary measure on a locally compact Abelian (LCA) group G is a complex measure whose Fourier transform is of absolute value 1 everywhere. The problem of finding all such measures is known to be closely related to that of finding all invertible measures on G. In this paper, we find all unitary measures when G is the circle or a discrete group. If G is a torsion-free discrete group, the characterization generalizes a theorem of Bohr.

1. Introduction. Let G be a locally compact group, and let μ be a finite (complex-valued) regular Borel measure on G. Define $\mu^\sim$ by $\mu^\sim(E) = \mu(E^{-1})$ for all Borel sets E. We call μ unitary if $\mu^\sim = \mu^{-1}$ (i.e., $\mu^\sim * \mu = \mu * \mu^\sim = \delta_e$). If G is Abelian, this condition is equivalent to saying that $|\hat{\mu}(\delta)| = 1$ for every $\gamma \in \Gamma$, the dual group of G.

We investigate the problem of finding all unitary measures on a locally compact Abelian group. The key tools are results of J. Taylor [7] and the Arens-Royden theorem (one form of which is given as Proposition 4.1 of [7]). We obtain complete results when G is discrete (Theorem 5); in the particular case where G is also torsion-free, the answer has a nicer form (Theorem 4), and this result generalizes a theorem of Bohr [1]. We also obtain all unitary measures on the circle group T; this was the original aim of the paper. In view of Corollaries 4.6 and 4.7 of [7], these results give characterizations of the measures in $\mathfrak{M}(G)^\times$, the multiplicative group of invertible measures on G, when G is one of the above groups.

Before getting down to serious work, we make a few preliminary remarks. If μ is unitary, write $\mu = \mu_d + \mu_c$, where μ_d is discrete and μ_c is continuous. Then (see [2]) μ_d is also unitary, and so $\mu = \mu_d * (\delta_e + \mu_c * \mu_d^\sim)$. Thus classifying the unitary measures amounts to classifying the discrete ones and those which are (continuous + δ_e). We call the latter continuous unitary measures.

Next, if ν is a measure on ν satisfying $\nu^\sim = -\nu$ (for Abelian G, this $\Leftrightarrow$ $\hat{\nu}$ is purely imaginary), then $\exp \nu = \delta_e + \nu + (\nu * \nu)/2! + \cdots$ is unitary. (Conversely, if $\exp \nu$ is unitary, then $\nu^\sim = -\nu$.) Unitary measures of this form are precisely the ones lying in the connected component of the identity of $\mathfrak{M}(G)^\times$. We gener-

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ally regard these measures as trivial and look for others. Corollaries 4.6 and 4.7 of [7] show that if $\mu \in \mathfrak{M}(G)^{\times}$, then $\exists$ a measure $\nu_0 \in \mathfrak{M}(G)$ such that $(\exp \nu_0)^{\frown}(\gamma) = |\mu(\gamma)|$, $\forall \gamma \in \Gamma$; it is this result that makes the problem of classifying unitary measures equivalent to the problem of classifying elements of $\mathfrak{M}(G)^{\times}$.

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2. **Continuous unitary measures.** The main purpose of this section is to prove the following theorem for T.

Theorem 1. *If μ is a continuous unitary measure on T, $\mu = \exp \nu$ for some continuous ν.*

Proof. Theorem 3 of [7] says that there are measures $\mu_1, \cdots, \mu_n, \nu_0 \in \mathfrak{M}(T)$, topologies $\mathcal{J}_1, \cdots, \mathcal{J}_n$ on T (all at least as fine as the usual topology, and all making T into a locally compact group), and complex numbers $\lambda_1, \cdots, \lambda_n$ such that $\mu_j - \lambda_j \delta_e$ is absolutely continuous with respect to Haar measure on $(T, \mathcal{J}_j)$ and $\mu = \mu_1 * \cdots * \mu_n * \exp(\nu_0)$. But the only topologies on T satisfying the condition are the usual one and the discrete topology.(1) Hence $\mu = \mu_1 * \mu_2 * \exp(\nu_0)$, where μ_1 is discrete and $\mu_2 - \lambda_2 \delta_e$ is absolutely continuous. Write $\nu_0 = \nu_1 + \nu_2$, where ν_1 is discrete and ν_2 is continuous; then $\mu = (\mu_1 * \exp \nu_1) * (\mu_2 * \exp \nu_2)$; the first expression in parentheses is a discrete measure, and the second is (except for a multiple of δ_e) continuous. As $\mu - \delta_e$ is continuous, the first term must be a multiple of δ_e.

We have reduced the theorem to the following: If μ is unitary and $\mu - \delta_e$ is absolutely continuous, then $\mu = \exp \nu$ for some continuous ν. In fact, one can even pick ν absolutely continuous, as we shall see. We work in $\mathfrak{L}^1(T) \oplus \mathbf{C}\delta_e$; it suffices to show that μ is in the connected component of δ_e in $\mathfrak{M}(T)^{\times}$. If $\hat{\mu}(m) \neq -1$, $\forall m$, then $\delta_e + t(\mu - \delta_e)$ never has zero transform, and hence is invertible for all t. In general, $\lim_{k \to \infty} \hat{\mu}_0(k) = 0$; hence $\hat{\mu}$ is -1 on a finite set. Suppose $\hat{\mu}(k) = -1$ for $k = m_1, \cdots, m_l$; let σ be the measure given by $d\sigma(z) = i(z^{m_1} + \cdots + z^{m_l})\,dz$ (dz = Harr measure). Then $(\mu + t\sigma)^{\wedge}(k) = \hat{\mu}(k)$ except for $k = m_1, \cdots, m_l$; at those points, $(\mu + t\sigma)^{\wedge}(k) = -1 + it$. Hence μ and $\mu + \sigma$ are connected by a line of invertible measures and, as in the first part, $\mu + \sigma$ is in the connected component of δ_e. That proves the theorem.

The last part of the proof gives the following result.

(1) This follows from structure theory. Another proof: If $G = (T, \mathcal{J})$, then the identity map of $G \to T$ gives (by duality) a dense map of $\mathbf{Z}$ into $\hat{G}$. Hence $\hat{G}$ is monothetic. All monothetic groups are $\mathbf{Z}$ or compact [5, Theorem 2.3.2]; hence G is T or discrete.

Corollary 1. *Suppose that* G *is compact. Then if* $\mu = \delta_e + \mu_0$ *is unitary and* μ_0 *is absolutely continuous,* $\mu = \exp \nu$ *for some absolutely continuous* ν.

The argument is the same as above. Suppose $\hat{\mu} = -1$ at $\gamma_1, \cdots, \gamma_l$; one lets $d\sigma(x) = i(\gamma_1(x) + \cdots + \gamma_l(x))\,dx$ and reasons the same way. (The set where $\hat{\mu} = -1$ is finite because $\hat{\mu}_0$ is 0 at ∞.)

The argument of the first half of the theorem reduces the problem for general Abelian G to finding absolutely continuous unitary measures on various other groups (viz., G with finer topologies). But that still leaves a good deal of work in most cases.

A similar attack does, however, work in at least one other case.

Theorem 2. *If* μ *is a unitary measure on a discrete torsion group* G*, then* $\mu = e^{\nu}$ *for some* $\nu \in \mathfrak{M}(G)$.

Proof. Since $|\hat{\mu}| = 1$ and Γ is totally disconnected, $\hat{\mu}(\gamma) = e^{a(\gamma)}$ for some continuous function a. Now the Arens-Royden theorem says that $\mu = \exp(\nu)$ for some $\nu \in \mathfrak{M}(G)$.

3. Delta measures; unitary measures on torsion-free discrete groups. Theorem 2 implies that if $z \in T$ is of finite order, then δ_z, the point mass at z with mass 1, is of the form $\exp(\nu)$. Here is a converse.

Theorem 3. *If* $z \in T$ *has infinite order, then* $\delta_z \neq \exp(\nu)$ *for any measure* ν.

Proof. Let $z = e^{i\theta}$. Then $\hat{\delta}_z(n) = e^{-in\theta}$; if, therefore, $\exp(\nu) = \delta_z$, then $\hat{\nu}(n) = -in\theta + 2\pi i k_n, k_n \in \mathbf{Z}$. Let $K = \|\nu\|$, so that $K \geq |2\pi k_n - n\theta|$, and let p be an integer $> K$. Let $y = e^{-i\theta/p}$. Then $\exp(\nu/p) * \delta_y$ has a Fourier-Stieltjes transform whose range consists of pth roots of unity. Let the roots be $\omega_1, \cdots, \omega_p$; then the set S_j on which the transform is ω_j is, according to results on idempotent measures (see, e.g., [5, p. 61 ff.]), a union of arithmetic progressions (with finitely many exceptions). The idea in what follows is that the irrationality of θ/π makes it impossible for the S_j to be so orderly.

Let N be large enough so that the variations in the progressions have been ironed out by then; pick S_j so that it contains an infinite arithmetic progression, with common difference r, say. Since $|2\pi k_n - n\theta| < K$, we have $(n\theta - K)/2\pi \leq k_n \leq (n\theta + K)/2\pi$.

Replace n by $n + mr$; we get

$$\frac{(n+mr)\theta - K}{2\pi} \leq k_{n+mr} \leq \frac{(n+mr)\theta + K}{2\pi}, \quad \text{or} \quad \frac{mr\theta - 2K}{2\pi} \leq k_{n+mr} - k_n \leq \frac{mr\theta + 2K}{2\pi}.$$

But $k_n - k_{n+mr}$ is a multiple of p; thus there is a multiple of p between $(mr\theta - 2K)/2\pi$ and $(mr\theta + 2K)/2\pi$, $\forall m$. That means that for all m, there is an integer between $(mr\theta - 2K)/2\pi p$ and $(mr\theta + 2K)/2\pi p$, or that $-mr\theta/2\pi p$ is congruent (mod 1) to a number between $-2K/2\pi p > -1/\pi$ and $2K/2\pi p < 1/\pi$. As $mr\theta/2\pi p$ is irrational, this is impossible; in fact, the numbers $-mr\theta/2\pi p$ are dense.

Corollary 2. *If G is any locally compact group and $x \in G$ has infinite order, then δ_x is not an exponential.*

Proof. We may as well assume that G is discrete, since if $\delta_x = \exp \nu$ and $\nu = \nu_1 + \nu_2$, with ν_1 discrete and ν_2 continuous, then $\delta_x = \exp \nu_1$ also. As T is divisible, we can extend the map $\alpha : nx \mapsto e^{in}$ to a homomorphism (also called α) of G into T. If $\delta_x = \exp(\nu)$, then it is easily checked that $\delta_{\alpha(x)} = \exp(\alpha_x \nu)$, where $\alpha_x \nu(E) = \nu(\alpha^{-1}(E))$. But Theorem 3 makes this impossible.

Theorem 3 makes it possible to find all the unitary measures on any torsion-free discrete group.

Theorem 4. *Let G be discrete and torsion-free. Then every unitary measure μ on G is of the form $\delta_x * \exp \nu$, for some $x \in G$ and some measure ν on G with $\nu^\sim = -\nu$; moreover, x is uniquely determined by μ.*

Proof. It suffices to show that any function $f : \Gamma \to \mathbf{C}^x$ is homotopic to a character $X_x : \gamma \mapsto (x, \gamma)$, $x \in G$. For then we can choose $x \in G$ such that $\hat{\mu} \cdot X_{-x}$ is homotopic to the trivial map. Since $X_{-x} = \hat{\delta}_{-x}$, we can use Arens-Royden to show that $\mu * \delta_{-x} = \exp(\nu)$ for some ν, and the theorem follows. The uniqueness of x follows from Theorem 3.

Now we prove the homotopy result. Since Γ is compact, we can use Stone-Weierstrass to approximate f by a finite linear combination of characters, $f \approx \Sigma_{j=1}^n a_j \chi_{x_j} = g$, say, so that $\|f - g\|_\infty < \frac{1}{2} \inf_{\gamma \in \Gamma} |f(\gamma)|$. Then f and g are homotopic. Let Γ_0 be the intersection of the kernels of the χ_{x_j}. Then g is constant on Γ_0-cosets, and therefore we can define $\tilde{g}$ on Γ/Γ_0 by $\tilde{g}(x\Gamma_0) = g(x)$. $(\Gamma/\Gamma_0)^\wedge$ is the group generated by the x_j: therefore Γ/Γ_0 is isomorphic to a torus. But it is well known (see, e.g., [3, Theorem II. 7.1]) that the characters of T^m represent the homotopy classes of maps on T^m; hence $\tilde{g}$ is homotopic to a character $\tilde{\chi}$ of Γ/Γ_0. Pull $\tilde{\chi}$ back to Γ, getting χ; then g and χ are homotopic, as desired.

A corollary of the proof is

Corollary 3. *If Γ is a connected compact group, then $H^1(\Gamma, \mathbf{Z}) \cong G$. (The cohomology is Čech cohomology.)*

Proof. From [4], $H^1(\Gamma, \mathbf{Z}) \cong$ group of homotopy classes of maps from Γ to T. Since Γ is connected, G is torsion-free [5, Theorem 2.5.6]; the last part of the above proof does the rest. (The result is dual to one of Steenrod's: $H_1(G, T) \cong G$. See [6, Theorem 15]).

4. Unitary measures on arbitrary discrete groups and T. We have still not solved the problem of finding the unitary measures on T_d, since T_d has torsion elements. The following example shows that we can actually find other unitary measures besides δ-measures $*$exponentials.

Let $z_0 \in T$ have infinite order, and let $\mu = \frac{1}{2}(\delta_1 + \delta_{z_0} + \delta_{-1} - \delta_{-z_0})$. Then if n is even, $\hat{\mu}(n) = \frac{1}{2}(1 + z_0^{-n} + (-1)^{-n} - (-z_0)^{-n}) = 1$, while if n is odd, $\hat{\mu}(n) = z_0^{-n}$. Suppose now that $\mu = \delta_{z_1} * \exp(\nu)$ for some $z \in T$ and some measure ν. Then $\mu * \delta_{z_1^{-1}} = \exp(\nu)$. Let $\mu_0 = \frac{1}{2}(\mu * \delta_{z_1^{-1}}) * (\delta_1 + \delta_{-1})$, $\mu_1 = \frac{1}{2}(\mu * \delta_{z_1^{-1}}) * (\delta_1 - \delta_{-1})$. Then

$$\mu_0 + \mu_1 = \mu * \delta_{z_1^{-1}}, \quad \text{and} \quad \hat{\mu}_0(n) = \begin{cases} (\mu * \delta_{z_1^{-1}})\hat{}\,(n), & n \text{ even}, \\ 0, & n \text{ odd}. \end{cases}$$

Let $\alpha: T \to T$ take $z \mapsto z^2$, and let $\mu_0^* = \mu \circ \alpha$. Then $\hat{\mu}_0^*(n) = \hat{\mu}_0(2n) = z_1^{2n}$; also, $\hat{\mu}_0(2n) = \exp \hat{\nu}(2n) = \exp \hat{\nu} * (n)$. Hence $\delta_{z_1^{-2}} = \mu_0^* = \exp(\nu^*)$. It follows that z_1 has finite order in T.

On the other hand,

$$\hat{\mu}_1(n) = \begin{cases} (\mu * \delta_{z_1^{-1}})\hat{}\,(n) = (z_1 z_0^{-1})^n, & n \text{ odd}, \\ 0, & n \text{ even}, \end{cases}$$

and $z_2 = z_1 z_0^{-1}$ has infinite order. Also, $\hat{\mu}_1(n) = \exp \hat{\nu}(n)$ whenever n is odd. We can now use the same reasoning as in the proof of Theorem 3 to get a contradiction. Let $z_2 = e^{i\theta}$, and define K, p, and y as in Theorem 3. Then $\exp \nu/p * \delta_y * \frac{1}{2}(\delta_1 - \delta_{-1})$ has a Fourier-Stieltjes transform whose range consists of pth roots of unity and 0; the value is 0 on $2\mathbf{Z}$. Define the S_j as in Theorem 3, and the rest of the argument in Theorem 3 goes through. It follows that μ is not a point measure convolved with an exponential.

What this argument says is that Theorem 4 is false for $G = Z \oplus \mathbf{Z}/2\mathbf{Z}$. ($G$ is embedded in T as $z_0\mathbf{Z} \oplus \{-1, 1\}$.) Then $\Gamma \cong T \oplus \mathbf{Z}/2\mathbf{Z}$; $\hat{\mu}$ is 1 on one circle and z on the other. This construction generalizes.

Let G be any discrete group, and let G_1 be a finite subgroup (of order n, say). Then $\Gamma_1 = G_1^{\perp}$ is of index n; let $\Gamma_1, \Gamma_2, \cdots, \Gamma_n$ be the cosets. Let $\sigma_1, \sigma_2, \cdots, \sigma_n$ be the idempotent measures whose Fourier-Stieltjes transforms are the characteristic functions of $\Gamma_1, \cdots, \Gamma_n$ respectively. Let $x_1, \cdots, x_n$ be elements of G, and let ν be a measure on G with $\nu^{\sim} = -\nu$. Then $\mu = (\Sigma_{j=1}^n \delta_{x_j} * \sigma_j) * \exp \nu$ is unitary.

Theorem 5. *Every unitary measure on a discrete group G arises in this way.*

Proof. Let μ_1 be a unitary measure. As in Theorem 3, it suffices to show that $\hat{\mu}_1$ and $\hat{\mu}$ are homotopic for some $\mu = (\sum_{j=1}^n \delta_{x_j} * \sigma_j) * \exp(\nu)$. Again, as in Theorem 3, $\hat{\mu}_1$ is homotopic to a linear combination of finitely many characters: $\hat{\mu}_1 \sim \sum_{j=1}^m a_j \chi_{y_j} = f$, say. Let Γ_0 be the common kernel of $\chi_{y_1}, \cdots, \chi_{y_m}$; f gives $\bar{f}$ on Γ/Γ_0. We may assume from now on that $\Gamma_0 = \{1\}$, since from now on everything will be constant on Γ_0-cosets. Note that f is the transform of a measure; thus $f^{-1}\hat{\mu}_1 = (\exp \nu_0)\hat{}$.

Given our assumption, G is generated by $y_1, \cdots, y_m$; hence $G \cong \mathbf{Z}^k \oplus G_1$, where G_1 is finite of order n. Thus $\Gamma_1, \cdots, \Gamma_n$ are k-tori. Hence there are elements $x_1, \cdots, x_n$ such that χ_{x_j} and f are homotopic on Γ_j. It follows that if $\mu_0 = \sum_{j=1}^n \delta_{x_j} * \sigma_j$, then $\hat{\mu}_0^{-1} f$ is homotopic to the trivial map on each component. Hence $\hat{\mu}_0^{-1} f = (\exp \nu_1)\hat{}$, and the theorem follows.

In the case of T_d, G_1 is necessarily cyclic. A more careful analysis along the lines of the example shows that if one picks m as small as possible, then each x_j is determined modulo the torsion group of T_d.

Theorems 1 and 5 together determine all the unitary measures on T. As noted earlier, they also determine all the connected components of $\mathfrak{M}(T)^\times$. We state the result here for completeness.

Corollary 4. *Let μ be an invertible measure on T. Then there are an integer m, elements $z_1, \cdots, z_m \in T$, and a measure ν on T such that $\mu = \exp(\nu) * (\sum_{j=1}^m \delta_{z_j} * \sigma_{m,j})$, where $\sigma_{m,j}$ is the idempotent measure on T whose Fourier-Stieltjes transform is 1 on $m\mathbf{Z} + j$ and 0 elsewhere. If μ is unitary, μ can be expressed in the same form, but with $\tilde{\nu} = -\nu$.*

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